Last time:

had functions $f_n : [0,1] \to \mathbb{R}$

$$f_n(x) = n^2 x^n (1-x)$$

showed: $f_n(x) \to 0 \quad \forall x \in [0,1]$

$\Rightarrow$ $(f_n)$ converges pointwise to zero function!

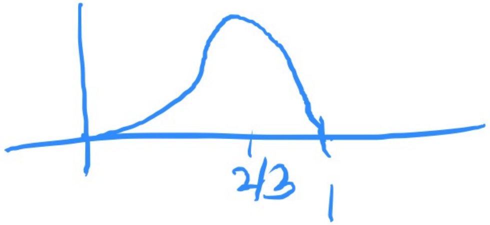
also found: $f_n$ reaches max. at $\frac{n}{n+1}$

$$f_n\left(\frac{n}{n+1}\right) = \underset{\substack{\downarrow \\ \infty}}{n} \cdot \underset{\substack{\downarrow \\ 1}}{\frac{n}{n+1}} \cdot \underset{\substack{\downarrow \\ 1/e}}{\frac{1}{(1+\frac{1}{n})^n}} \quad \text{for } n \to \infty$$

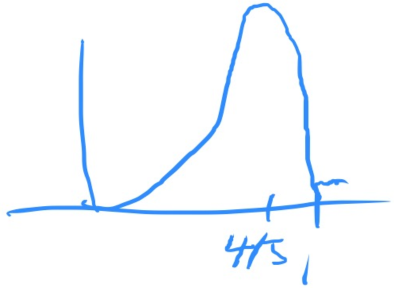
$\Rightarrow$ does NOT converge uniformly to zero function.

sketches of $f_n$'s

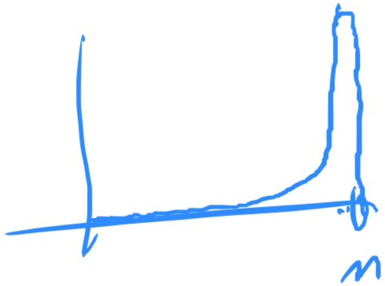
$f_2$




$f_4$




large $n$



# Ch. 25 More on uniform convergence

Behavior of uniformly converging functions under integration:

we will use following properties of integrals (to be proved later)

(a) If $g(x) \leq h(x)$ for all $a \leq x \leq b$

$$\Rightarrow \int_a^b g(x)\,dx \leq \int_a^b h(x)\,dx$$

(b) $$\int_a^b g(x)\,dx \leq \int_a^b |g(x)|\,dx$$

(c) continuous functions are integrable.

**Theorem** Assume $(f_n) \to f$ uniformly on $[a,b]$

$$\Rightarrow \lim \int_a^b f_n(x)\,dx \to \int_a^b f(x)\,dx$$

**Proof** need to estimate

should be $< \varepsilon$ $\to$ $$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| = \left| \int_a^b f_n(x) - f(x)\,dx \right|$$

$$\leq \int_a^b |f_n(x) - f(x)|\,dx$$

$f_n \to f$ uniformly $\Rightarrow$ for any $\tilde{\varepsilon} > 0$ can find $N$ s.t. $\boxed{|f_n(x) - f(x)| < \tilde{\varepsilon}}$ for all $x \in [a,b]$ all $n > N$

$$\leq \int_a^b \tilde{\varepsilon}\,dx = \tilde{\varepsilon}(b-a) = \varepsilon \Rightarrow \boxed{\tilde{\varepsilon} = \frac{\varepsilon}{b-a}}$$

let $\varepsilon > 0$ take $\tilde{\varepsilon} = \dfrac{\varepsilon}{b-a}$

$\Rightarrow \exists N$ s.t. if $n > N$ we have

$$|f_n(x) - f(x)| < \tilde{\varepsilon} = \frac{\varepsilon}{b-a}$$

$$\Rightarrow \left| \int f_n(x) - f(x)\, dx \right| < (b-a)\tilde{\varepsilon} = (b-a)\frac{\varepsilon}{(b-a)} = \varepsilon$$

by previous pag

# Cauchy Sequences for Functions

Recall: Cauchy sequences define convergence of a sequence without having to know the limit

can also be done for uniformly converging functions

**Def.** A sequence of functions $(f_n)$ on $S \subset \mathbb{R}$ is uniformly Cauchy if

for each $\varepsilon > 0$ we can find an $N$ s.t.

$|f_n(x) - f_m(x)| < \varepsilon$ for all $x$ in $S$
for all $n, m > N$.

**Theorem** Assume $(f_n)$ is uniformly Cauchy

$\Rightarrow$ $\exists$ function $f$ on $S$ such that $f_n \to f$ uniformly.

proof **Step 1** Define function $f$

Fix $x_0 \in S$

**claim:** $(f_n(x_0))_n$ is a Cauchy sequence of numbers!

proof. given $\varepsilon > 0$ $\exists N$ s.t. $|f_n(x) - f_m(x)| < \varepsilon$ $\forall n, m > N$ $\forall x \in S$

this is true, in particular, also for $x = x_0$

$\Rightarrow$ claim

$\Rightarrow$ $(f_n(x_0))$ converges to a number

define: $f(x_0) = \lim_{n \to \infty} f_n(x_0)$

## step 2

claim $f_n \to f$ uniformly

let $\varepsilon > 0$

$\exists N > 0$ s.t. $\boxed{\begin{array}{l} |f_n(x) - f_m(x)| < \varepsilon/2 \\ \text{for all } n, m > N \\ \text{for all } x \text{ in } S. \end{array}}$

$\Rightarrow f_n(x) \in (f_m(x) - \varepsilon/2,\ f_m(x) + \varepsilon/2) \quad \forall n > N$

$\Rightarrow f(x) = \lim_{n\to\infty} f_n(x) \in [f_m(x) - \varepsilon/2,\ f_m(x) + \varepsilon/2] \ni f(x)$

$$\Rightarrow |f_n(x) - f(t)| \le |f_n(x) - f_m(x) + f_m(x) - f(x)|$$

$$\le \underbrace{|f_n(x) - f_m(x)|}_{< \varepsilon/2} + \underbrace{|f_m(x) - f(x)|}_{\le \varepsilon/2}$$

$$< \varepsilon \qquad \forall n > N \ \checkmark$$

Find examples for uniform convergence!

Recall: A series $\sum_{k=0}^{\infty} a_k$ converges to $a$

$\Longleftrightarrow$ if for partial sums $S_n = \sum_{k=0}^{n} a_k$

we have $\lim_{n\to\infty} S_n = a$

Same for functions:

Let $g_k : S \longrightarrow \mathbb{R}$ $k = 0, 1, 2, 3, \ldots$

Let $f_n(x) = \sum_{k=0}^{n} g_k(x)$

Then we say

$\sum_{k=0}^{\infty} g_k(x)$ converges uniformly to $f(x)$

if $f_n \to f$ uniformly.

# Weierstraß M-test

Let $(M_k)$ be a sequence of nonzero real numbers such that $\sum_{k=0}^{\infty} M_k < \infty$ ($\Rightarrow \sum M_k$ converges)

Assume we have functions $g_k$ s.t.

$$|g_k(x)| \leq M_k \quad \forall x \in S$$

$$\Rightarrow \sum_{k=0}^{\infty} g_k(x) \text{ converges uniformly to a function } f(x)$$

Proof. need to show: if $f_n(x) = \sum_{k=0}^{n} g_k(x)$

$\Rightarrow f_n \to f$ uniformly

let $\varepsilon > 0$ As $\sum M_k < \infty$ it follows that

$\sum M_k = M$ for some $M$

$\Rightarrow s_m = \sum_{k=0}^{m} M_k \Rightarrow (s_m)$ is a Cauchy sequence.

$\Rightarrow \exists N$ s.t. $|S_n - S_m| < \varepsilon \quad \forall n, m > N$

say $n \geq m$

$$\Rightarrow \quad S_n - S_m = \sum_{k=0}^{n} M_k - \sum_{k=0}^{m} M_k$$

$$= \sum_{k=m+1}^{n} M_k < \varepsilon$$

$$\Rightarrow |f_n(x) - f_m(x)| = \left| \sum_{k=0}^{n} g_k(x) - \sum_{k=0}^{m} g_k(x) \right|$$

$$= \left| \sum_{k=m+1}^{n} g_k(x) \right|$$

$\Delta$-inequality

$$\leq \sum_{k=m+1}^{n} |g_k(x)| \leq \sum_{k=m+1}^{n} M_k < \varepsilon$$

$\Rightarrow$ $(f_n)$ is a uniform Cauchy sequence. $\Rightarrow$ $f_n$ converges uniformly to a function $f$.